Pricing and Hedging of VolDex Futures and Options

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This version: July 24, 2017

Abstract

The VolDex is an indicator of at-the-money implied volatility, defined by a linear function of an ATM option price. We price and hedge futures and options on the VolDex, under a jump-diffusive model of instantaneous variance dynamics that allow for (1) consistent modeling of VolDex and the underlying (2) flexibility to generate various shapes for the implied volatility skews of the underlying and of VolDex, and (3) tractable computations.

1 Introduction

Given a tenor τ, let $F^*_{t,\tau}$ denote the time-t forward price for time-$(t+\tau)$ delivery of a specified underlying.

Define

$$\mathbb{V}_t := \mathbb{V}_{t,\tau} := 100\sqrt{2\pi}\ \frac{P_{t,\tau}e^{r\tau}}{F^*_{t,\tau}\sqrt{\tau}} \tag{1.1}$$

where $P_{t,\tau}$ denotes the time-t price of the at-the-money put option that pays $(F^*_{t,\tau} - F^*_{t+\tau,0})^+$ at time-$(t+\tau)$, and r is the interest rate, assumed constant.

Fixing an expiry date T, we price and hedge futures and options on $\mathbb{V}_T$, which may be rewritten as

$$\mathbb{V}_T = \mathbb{V}_{T,\tau} = ae^{r\tau}\ \frac{P_{T,\tau}}{F_{T,\tau}} \tag{1.2}$$

where $a := 100\sqrt{2\pi/\tau}$, and $F_{t,\tau}$ denotes the time-t forward price for time-$(T+\tau)$ delivery of the underlying.

In the application that motivates us, $\mathbb{V}_t$ models the time-t spot level of the *VolDex*, an indicator of ATM volatility, defined such that τ is 1 month, and the underlying asset is the SPY ETF. In practice, the VolDex interpolates the $\mathbb{V}_t$ from the prices of near-ATM puts at weekly expiries near $t+\tau$, as detailed in [5] and summarized in Appendix A. In practice, moreover, the exchange-listed puts on SPY are American-style, but in frictionless markets with zero interest rates, American puts have zero early-exercise premium, hence negligible early-exercise premium in environments of sufficiently low rates.

*The author acknowledges the support of the ISE and the Nasdaq, and the research assistance of Hyomin Choi. VolDex is a registered trademark of NationsShares, a division of NShares LLC. All other company and product names mentioned are used only for identification purposes and may be trademarks or registered trademarks of their respective companies.

Let ϕ denote the time-0 futures price for time-T delivery of $\mathbb{V}_T$.

Let ψ or ψ_K denote the time-0 price of a T-expiry K-strike call on $\mathbb{V}_T$.

2 Model

2.1 Model selection

The following criteria guide our choice of a dynamic model.

First is the capability to model not only $\mathbb{V}_t$ but also the underlying F_t in a consistent way. This is important particularly because we intend to use options on F in some of our strategies to hedge $\mathbb{V}$ contracts. For this purpose it is not satisfactory to postulate "directly" a model on the dynamics of $\mathbb{V}_t$ in the absence of consistent dynamics for F_t. So the question becomes, how to model F.

The second criterion is the flexibility to generate a variety of shapes for the implied volatility skews of F *and* $\mathbb{V}$; of particular interest (in equity markets) is a downward-sloping F skew together with an *upward*-sloping $\mathbb{V}$ skew. Regarding the latter point, it is empirically observed that the implied volatility skews for various types of volatilty/variance products tend to increase as a function of strike. To be precise, given ψ_K, the implied volatility of $\mathbb{V}$ at strike K and expiry T is defined to be the σ_{imp} such that

$$\psi_K = e^{-rT}(\phi N(d_1) - KN(d_2)), \qquad d_{1,2} := d_{+,-} := \frac{\log(\phi/K)}{\sigma_{\text{imp}}\sqrt{T}} \pm \frac{\sigma_{\text{imp}}\sqrt{T}}{2} \tag{2.1}$$

where N denotes the standard normal CDF. The upward-slope phenomenon – in which σ_{imp} increases as K increases – is difficult to capture using, for instance, the standard Heston model of stochastic volatility, which tends to generate *downward*-sloping skews for volatility/variance contracts. So our attention turns to modeling the F dynamics using a Heston model augmented with upward jumps in instantaneous variance, with sufficient flexibility to generate various slopes for implied volatility of $\mathbb{V}$.

The third criterion is tractability, in the sense that futures and options on $\mathbb{V}$ have prices given by explicit formulas numerically computable using single or at most double integration. This criterion is satisfied if the jumps have exponential distributions.

The resulting Heston+jump model meets all three criteria. The model postulates that the underlying futures price F and its instantaneous variance V follow (under a martingale measure $\mathbb{P}$) the dynamics

$$\begin{aligned} \mathrm{d}F_t &= \sqrt{V_t}F_t\mathrm{d}B_t \\ \mathrm{d}V_t &= \kappa(\theta - V_t)\mathrm{d}t + \sigma\sqrt{V_t}\mathrm{d}W_t + \mathrm{d}J_t \end{aligned} \tag{2.2}$$

where J is a compound Poisson process, independent of B and W, with jump-arrival intensity λ and exponentially distributed jumps with mean μ, and B and W are Brownian motions with $\mathrm{d}\langle B, W\rangle = \rho\mathrm{d}t$, so ρ is the correlation between the diffusive fluctuations in price and volatility.

The $\theta > 0$ is the level toward which the instantaneous variance V reverts (in the absence of jumps), and the $\kappa > 0$ is the rate of reversion. The σ controls the volatility of the (diffusive part of the) V dynamics. The instantaneous variance has long-run mean $\theta + \lambda\mu/\kappa$, taking account of the presence of jumps.

This model is nested in the SVJJ specification of Duffie-Pan-Singleton ([2], which priced simple options on F, but here we are ultimately interested in options on $\mathbb{V}$, which are *compound* options on F).

2.2 Characteristic functions

The distributions of the variables in the model are described by their characteristic functions.

By [2], the time-T conditional characteristic function of $\log(F_{T+\tau}/F_T)$ is $z \mapsto f_X(z, V_T)$ where

$$f_X(z,v) = \exp(A_X(z) + B_X(z)v) \tag{2.3}$$

where, excluding degenerate cases,

$$\begin{aligned}
A_X(z) &:= \frac{\kappa\theta}{\sigma^2}\left[(\kappa_* - \gamma)\tau - 2\log\left(1 + \frac{\kappa_* - \gamma}{2\gamma}(1 - e^{-\gamma\tau})\right)\right] \\
&\quad + \lambda\tau\Big(\frac{\gamma + \kappa_*}{\gamma + \kappa_* + \mu w} - 1\Big) - \frac{2\lambda\mu w}{\gamma^2 - (\kappa_* + \mu w)^2}\log\left(1 - \frac{(\gamma - \kappa_* - \mu w)(1 - e^{-\gamma\tau})}{2\gamma}\right) \\
B_X(z) &:= \frac{-w(1 - e^{-\gamma\tau})}{2\gamma e^{-\gamma\tau} + (\gamma + \kappa_*)(1 - e^{-\gamma\tau})} \\
\kappa_* &:= \kappa - i\rho\eta z, \qquad \gamma := \sqrt{\kappa_*^2 + \sigma^2 w}, \qquad w := iz + z^2.
\end{aligned} \tag{2.4}$$

By Duffie-Garleanu ([1] in the $q \to 0$ limit), the (time-0) characteristic function of V_T, excluding degenerate cases, is $u \mapsto f_V(u, V_0)$ where

$$f_V(u,v) = \exp(A_V(u) + B_V(u)v), \tag{2.5}$$

where

$$\begin{aligned}
A_V(u) &:= -\frac{2\kappa\theta}{\sigma^2}\log\Big(1 + \frac{\sigma^2 iu}{2\kappa}(e^{-\kappa T} - 1)\Big) \\
&\quad + \frac{2\lambda\mu}{2\kappa\mu - \sigma^2}\log\Big(1 + \frac{iu(\sigma^2 - 2\kappa\mu)(e^{-\kappa T} - 1)}{2\kappa(1 - iu\mu)}\Big) \\
B_V(u) &:= \frac{2\kappa iu}{2\kappa e^{\kappa T} + \sigma^2 iu(1 - e^{\kappa T})}.
\end{aligned} \tag{2.6}$$

In our terminology, *vega* will mean the sensitivity (partial derivative) of the value of a contract with respect to *instantaneous variance* V_0. The symbol $\mathcal{V}$ will denote vega, and its subscript will designate the contract in question. The symbol $\mathcal{V}^*$ will denote *discrete* vega, the sensitivity of the contract value with respect to bumping instantaneous variance up by δ for some designated $\delta > 0$.

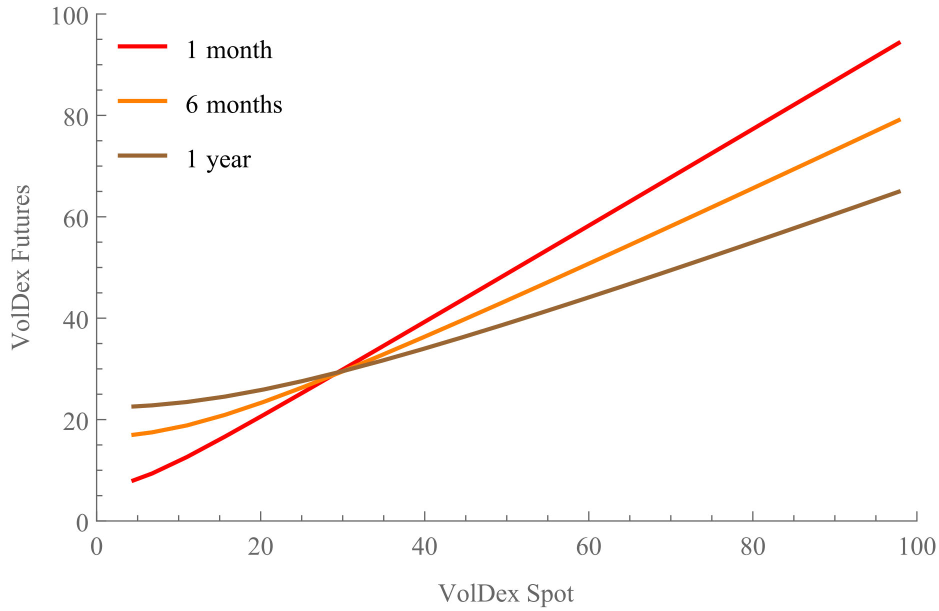
3 Pricing

3.1 VolDex spot

Let $\mathbb{E}_t$ denote time-t conditional $\mathbb{P}$-expectation. Then VolDex spot satisfies

$$\mathbb{V}_t = a\mathbb{E}_t\Big(1 - \frac{F_{t+\tau}}{F_t}\Big)^+ = G(V_t) \tag{3.1}$$

Figure 1: VolDex Spot vs. Futures



where the ATM option-pricing function G is given by

$$G(v) = a + \frac{a}{2\pi} \int_{-\infty-\alpha i}^{\infty-\alpha i} \frac{f_X(z,v)}{-z(z+i)} \mathrm{d}z \quad \text{for } 0 < \alpha < 1. \tag{3.2}$$

For instance, one can take the midpoint $\alpha = 1/2$. Indeed, α in a range outside $(0,1)$ can also be chosen, if the leading term (a) is multiplied by the appropriate adjustment; see Lewis [4] and Lee [3].

3.2 Futures

The time-0 VolDex futures price for time-T delivery follows from (3.2):

$$\phi = \mathbb{E}G(V_T) = a + \frac{a}{\pi} \int_{0-\alpha i}^{\infty-\alpha i} \mathrm{Re}\, \frac{\mathbb{E}e^{A_X(z)+B_X(z)V_T}}{-z(z+i)} \mathrm{d}z = a + \frac{a}{\pi} \int_{0-\alpha i}^{\infty-\alpha i} \mathrm{Re}\, \frac{e^{A_X(z)} f_V(-iB_X(z), V_0)}{-z(z+i)} \mathrm{d}z. \tag{3.3}$$

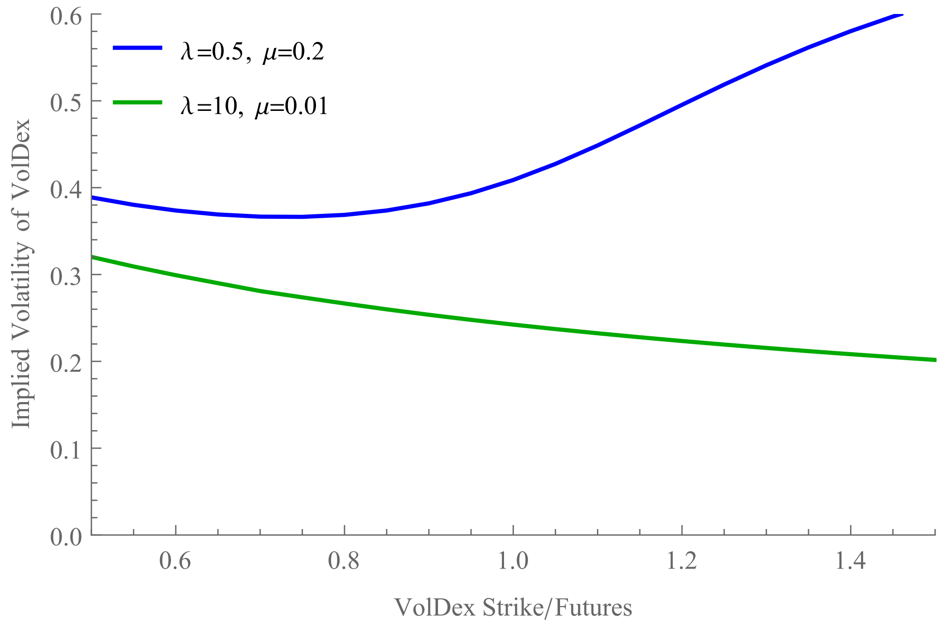
Differentiating with respect to V_0 gives the vega of the futures contract:

$$\mathcal{V}_\phi = \frac{a}{\pi} \int_{0-\alpha i}^{\infty-\alpha i} \mathrm{Re}\, \frac{e^{A_X(z)} B_V(-iB_X(z)) f_V(-iB_X(z), V_0)}{-z(z+i)} \mathrm{d}z \tag{3.4}$$

for arbitrary $\alpha \in (0,1)$.

Figure 1 plots the VolDex futures at three expiries, against VolDex spot. The relationship between spot and futures depends on the dynamics of the model; here we take $\kappa = 1$, $\theta = 0.04$, $\sigma = 0.2$, $\rho = -0.5$, $\lambda = 0.2$, $\mu = 0.5$. At the short (1 month) expiry, the futures are approximately "delta one" with respect to spot; but at the longer expiries, the effect of mean reversion flattens the sensitivity of futures to spot.

Figure 2: Implied Volatility-of-VolDex



3.3 Options

Consider a call on VolDex that pays $(\mathbb{V}_T - K)^+$ at time T. In the trivial case where $K < G(0)$, the call is sure to finish in the money, and has time-0 price $\psi = (\phi - K)e^{-rT}$, and time-0 vega $\mathcal{V}_\psi = \mathcal{V}_\phi e^{-rT}$.

So consider only the case $K \geq G(0)$, and define v_K by

$$G(v_K) = K. \tag{3.5}$$

From (3.2), the complex Fourier transform of $v \mapsto (G(v) - K)^+$ is, for $\operatorname{Im} u > 0$,

$$\begin{aligned} g(u) &= \int_{-\infty}^{\infty} \left(\frac{a}{2\pi} \int_{-\infty-\alpha i}^{\infty-\alpha i} \frac{e^{A_X(z)+B_X(z)v} e^{iuv} \mathbf{1}_{v>v_K}}{-z(z+i)} \mathrm{d}z + (a-K)e^{iuv}\mathbf{1}_{v>v_K} \right) \mathrm{d}v \\ &= (K-a)\frac{e^{iuv_K}}{iu} + \frac{a}{2\pi} \int_{-\infty-\alpha i}^{\infty-\alpha i} \frac{e^{A_X(z)} e^{(B_X(z)+iu)v_K}}{z(z+i)(B_X(z)+iu)} \mathrm{d}z \end{aligned} \tag{3.6}$$

where $0 < \alpha < 1$. The interchange in (3.6) is valid because $\operatorname{Re} A_X(z)$ is bounded above and $\operatorname{Re} B_X(z) < 0$.

For $0 < \beta < 1/\mu$, both $x \mapsto f_V(x - \beta i)$ and $x \mapsto g(-x + \beta i)$ are in $L^2(\mathbb{R})$, so by Plancherel, we have the call pricing formula

$$\psi = e^{-rT}\mathbb{E}(G(V_T) - K)^+ = e^{-rT} \int p(v)(G(v) - K)^+ \mathrm{d}v = \frac{1}{\pi e^{rT}} \int_{0-\beta i}^{\infty-\beta i} \operatorname{Re}\left[f_V(u, V_0) g(-u) \right] \mathrm{d}u. \tag{3.7}$$

where p is the density of V_T, which exists, assuming that $2\kappa\theta > \sigma^2$. For instance, one can take the midpoint, $\beta = 1/(2\mu)$ of the interval of validity $(0, 1/\mu)$.

By put-call parity, the put price equals the corresponding call price, plus $(K - \phi)e^{-rT}$.

Taking the V_0-derivative gives the vega of the call or put contract:

$$\mathcal{V}_\psi = \frac{1}{\pi e^{rT}} \int_{0-\beta i}^{\infty-\beta i} \mathrm{Re}\left[B_V(u) f_V(u, V_0) g(-u)\right] \mathrm{d}u. \tag{3.8}$$

Figure 2 plots the implied volatility of VolDex options, for strikes ranging from 50% to 150% of the ATM strike, and expiry 6 months. The top curve uses $V_0 = 0.14$ and the Figure 1 parameters, including $\lambda = 0.5$ and mean jump size 20 percentage points in V. The bottom curve uses the same parameters, except $\lambda = 10$ and a smaller mean jump size 1 percentage point. The larger mean jump size μ induces a volatility-of-VolDex skew that slopes upward ATM, whereas the smaller μ generates a downward slope.

4 Hedging

In order to hedge VolDex contracts, consider two possible hedging instruments: denote by ϕ the T-expiry VolDex futures (to hedge T-expiry VolDex options), and denote by C a volatility-sensitive $(T+\tau)$-expiry contract on F (to hedge T-expiry VolDex futures or options).

We leave flexible, the choice of whether the contract C is a call/put on F at some strike K_F, whose vega and discrete vega are respectively

$$\begin{aligned} \mathcal{V}_C &= \frac{a}{\pi e^{r(T+\tau)}} \int_{0-\alpha i}^{\infty-\alpha i} \mathrm{Re}\, \frac{B_X(V_0) f_X(z, V_0) e^{(1-iz)\log(K_F/F_0)}}{-z(z+i)} \mathrm{d}z \\ \mathcal{V}_C^* &= \frac{a}{\pi \delta e^{r(T+\tau)}} \int_{0-\alpha i}^{\infty-\alpha i} \mathrm{Re}\, \frac{(f_X(z, V_0+\delta) - f_X(z, V_0)) e^{(1-iz)\log(K_F/F_0)}}{-z(z+i)} \mathrm{d}z, \end{aligned} \tag{4.1}$$

or whether C is instead a combination/mixture of calls/puts on F, for instance a *log contract*, which pays $-2\log(F_{T+\tau}/F_0)$, and whose vega and discrete vega are

$$\mathcal{V}_C = \mathcal{V}_C^* = \frac{1 - e^{-\kappa(T+\tau)}}{\kappa e^{r(T+\tau)}}. \tag{4.2}$$

4.1 Hedging VolDex futures

To vega-neutralize a short position in a VolDex futures contract, go long

$$\frac{\mathcal{V}_\phi}{\mathcal{V}_C} \tag{4.3}$$

contracts on the underlying.

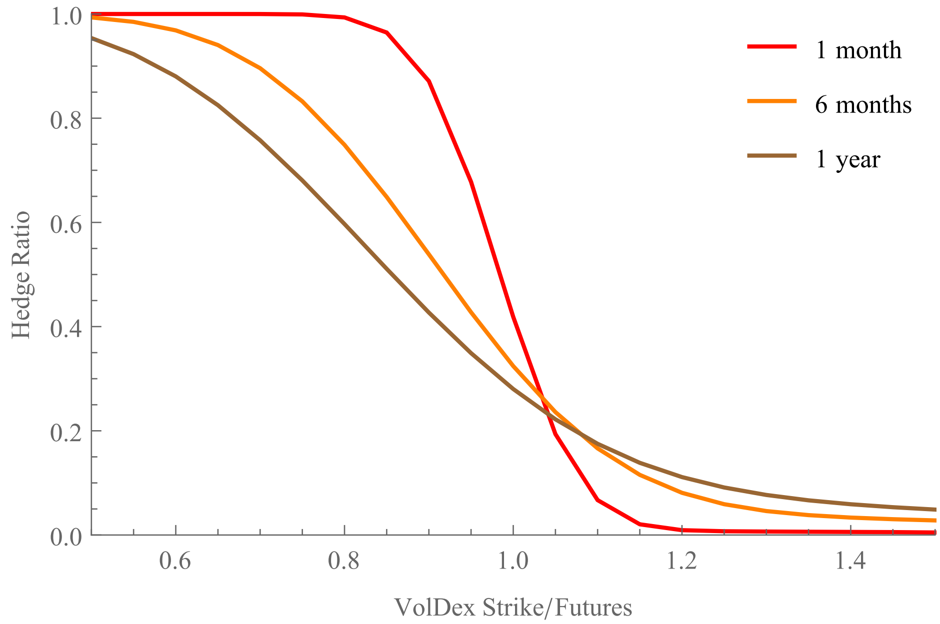
4.2 Hedging a VolDex option

4.2.1 Using VolDex futures

To vega-neutralize a short position in a VolDex call, go long

$$\frac{\mathcal{V}_\psi}{\mathcal{V}_\phi} \tag{4.4}$$

Figure 3: Hedge Ratios for VolDex Calls vs. VolDex Futures



VolDex futures contracts. Figure 3 plots (4.4) for strikes ranging from 50% to 150% of the ATM strike, using the Figure 1 parameters and $V_0 = 0.14$. The behavior of this hedge ratio resembles the familiar delta of a call option, decreasing from 1 to 0 as the strikes increase. Near expiration, the decrease occurs sharply near ATM, but far from expiration, the decrease takes place gradually over a wide range of strikes.

4.2.2 Using contracts on the underlying

Instead of using VolDex futures, we can use contracts on the underlying as the hedging instrument.

In this case, to vega-neutralize a short position in a VolDex call, go long

$$\frac{\mathcal{V}_\psi}{\mathcal{V}_C} \tag{4.5}$$

contracts on the underlying (SPY, in our primary application).

4.2.3 Using VolDex futures and contracts on the underlying

In the no-jump case ($\lambda\mu = 0$), vega neutralization (combined with delta neutralization) in continuous time produces a perfect hedge. In the presence of jumps in V, however, vega/delta neutralization does not imply perfect hedging of jump risk. In that case, using *two* volatility-sensitive hedging instruments can, in addition to neutralizing vega, also neutralize the impact of jumps of a designated size δ. Define the discrete vegas

$$\mathcal{V}_C^* := \frac{C(V_0+\delta) - C(V_0)}{\delta}, \qquad \mathcal{V}_\phi^* := \frac{\phi(V_0+\delta) - \phi(V_0)}{\delta}, \qquad \mathcal{V}_\psi^* := \frac{\psi(V_0+\delta) - \psi(V_0)}{\delta}, \tag{4.6}$$

where C, ϕ, ψ are regarded as functions of instantaneous variance. Solving for the quantities (q_ϕ, q_C) in

$$\begin{aligned} q_\phi \mathcal{V}_\phi + q_C \mathcal{V}_C &= \mathcal{V}_\psi \\ q_\phi \mathcal{V}^*_\phi + q_C \mathcal{V}^*_C &= \mathcal{V}^*_\psi \end{aligned} \tag{4.7}$$

yields the hedging portfolio consisting of positions

$$\begin{aligned} q_\phi &= \frac{\mathcal{V}_\psi \mathcal{V}^*_C - \mathcal{V}^*_\psi \mathcal{V}_C}{D} && \text{VolDex futures} \\ q_C &= \frac{\mathcal{V}^*_\psi \mathcal{V}_\phi - \mathcal{V}_\psi \mathcal{V}^*_\phi}{D} && \text{contracts on the underlying} \end{aligned} \tag{4.8}$$

provided that

$$D := \mathcal{V}_\phi \mathcal{V}^*_C - \mathcal{V}_C \mathcal{V}^*_\phi \neq 0. \tag{4.9}$$

The strategy (4.8) theoretically hedges vega and δ-jumps, but practical considerations (transaction costs, model risk) may motivate restricting the position sizes of the hedging instruments. In particular, suppose one constrains the positions in both hedging instruments to be nonnegative, while maintaining the vega hedge. In that case,

$$\begin{aligned} &0 \vee q_\phi \wedge \frac{\mathcal{V}_\psi}{\mathcal{V}_\phi} && \text{VolDex futures} \\ &0 \vee q_C \wedge \frac{\mathcal{V}_\psi}{\mathcal{V}_C} && \text{contracts on the underlying} \end{aligned} \tag{4.10}$$

would be the hedging portfolio nearest to (4.8).

5 Conclusions

In a computationally tractable model consistently encompassing the underlying and VolDex dynamics, we have solved analytically for prices and hedges of VolDex futures and options. Numerical experiments show: an implied volatility-of-VolDex skew that has the flexibility to slope upward or downward depending on the parameters; a VolDex futures/spot relationship that exhibits a range of sensitivities, nearly delta-one for small expiries and flatter for long expiries; and a VolDex calls/futures hedge ratio that behaves, as a function of strike, analogously to a standard delta profile, by decreasing from 1 to 0, gradually for longer expiries and sharply for shorter expiries.

A Appendix: VolDex construction from weekly options

In practice, the VolDex as specified in [5] interpolates $\mathbb{V}_{T,\tau}$ as the square root of

$$\frac{p\tau_{\text{frt}}\mathbb{V}^2_{T,\tau_{\text{frt}}} + (1-p)\tau_{\text{sct}}\mathbb{V}^2_{T,\tau_{\text{sct}}}}{\tau}$$

where $\{\tau_{\text{frt}}, \tau_{\text{sct}}\}$ denote, respectively, the lengths of time remaining from T until the weekly Friday SPY option expiries immediately prior (or equal) to, and subsequent to, date $T+\tau$; and $p := (\tau_{\text{sct}}-\tau)/(\tau_{\text{sct}}-\tau_{\text{frt}})$, while the $P_{T,\tau}$ is interpolated as

$$qP_{T,\tau,K_{\text{otm}}} + (1-q)P_{T,\tau,K_{\text{itm}}}$$

where $P_{T,\tau,K}$ is the time-T price of the put option that pays $(K - F^*_{T+\tau,0})^+$ at time $T+\tau$; and $\{K_{\text{otm}}, K_{\text{itm}}\}$ denote, respectively, the listed integer strikes immediately below and above the forward price $F^*_{T,\tau}$; and $q := (K_{\text{itm}} - F^*_{T,\tau})/(K_{\text{itm}} - K_{\text{otm}})$.

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